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Irreducible representations of quantum linear groups of type $A_{1|0}$ [☆]

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Introduction

A Hecke symmetry is an invertible operator defined on the tensor square of a finite-dimensional vector space, which satisfies the (quantized) Yang–Baxter equation and the Hecke equation $(x + 1)(x - q) = 0$. Given such a matrix, one can construct a Hopf algebra which is "the function algebra" over an (algebraic) matrix quantum group. One is interested in the representations of this quantum group, that is, the comodules over the Hopf algebra.

It turns out that the dimension of the vector space on which the Hecke symmetry is defined does not play a significant role, but it is rather a certain "inner" characteristics of the Hecke symmetry, the birank – a pair of non-negative integers – which is relevant.

A Hecke symmetry is called even if its birank is of the form $(r, 0)$ and odd if its birank is of the form $(0, r)$. In these cases, the representation category of the corresponding quantum group is like the representation category of the matrix group $GL(r)$, it is semisimple and generally does not depend on the Hecke symmetry but only on the birank.

If the Hecke symmetry is neither even nor odd the representation category of the corresponding quantum group seems to be similar to the representation category of the super group $GL(r|s)$. This problem has not been settled yet. The aim of this paper is to classify

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irreducible representation of the matrix quantum group corresponding to a Hecke symmetry of birank $(2, 1)$. Such a quantum group is usually called matrix quantum group of type $(1, 0)$.

In this work we show that irreducible representations of a quantum group of type $(1, 0)$ can be indexed by tuples (m, n, p) of integers with $m \geqslant n$. We exhibit the basic decomposition rules of the tensor products of these representations and compute their dimensions.

Recall that representations of a quantum group are by definition comodules over the corresponding Hopf algebra ("of functions"). Using the Koszul complex we construct for each tuple (m, n, p), $m \geqslant n$ a comodule and prove that they are simple. The difficulty here is that the comodule category is not semisimple. Our main technique is based on the theory of Hopf algebras with integral (co-Frobenius Hopf algebras). For such Hopf algebras there is a special class of simple comodules that split in any comodule. We show that our Hopf algebra is co-Frobenius and use a criterion for a comodule to be splitting in any comodule (which is equivalent to being injective and projective).

In Kac's theory of representation of Lie superalgebras irreducible representations that split in any representations are called typical [10]. Notice that our method when applied to the algebraic supergroup $GL(2|1)$ provides a new approach to the problem of classification of irreducible representations of this supergroup.

From the Hopf algebra theory view-point, our result provides a non-trivial example of a non-cosemisimple infinite-dimensional co-Frobenius Hopf algebra, whose simple comodules are fully classified by the utilization of the integral.

1. Quantum groups of type A and their representations

Throughout this paper k will be an algebraically closed field of characteristic zero, $q \in k^\times$ is a fixed element, which is not a root of the unit element of degree greater than 1. For simplicity we shall usually denote an isomorphism by the equation sign "$=$" and a direct sum by the plus sign "$+$".

1.1. Hecke symmetries and the associated quantum groups

Let V be a vector space over k, of finite dimension d. Let $R : V \otimes V \to V \otimes V$ be an invertible operator. R is called a *Hecke symmetry* if the following conditions are fulfilled:

(i) $R_1 R_2 R_1 = R_2 R_1 R_2$, where $R_1 := R \otimes \mathrm{id}_V$, $R_2 := \mathrm{id}_V \otimes R$,
(ii) $(R + 1)(R - q) = 0$ for some $q \in k^\times$,
(iii) the half adjoint to R, $R^\sharp : V^* \otimes V \to V \otimes V^*$, $\langle R^\sharp(\xi \otimes v), w\rangle = \langle \xi, R(v \otimes w)\rangle$, is invertible.

Throughout this work we will assume that q *is not a root of unity other than the unity itself*.

Fix a basis $x_1, x_2, \ldots, x_d$ of V. Then R can be given in terms of a matrix, also denoted by R, $R(x_i \otimes x_j) = x_k \otimes x_l R^{kl}_{ij}$ (here and further on we adopt the convention of summing up over the indices that appear both in upper and lower places). The matrix ${R^\sharp}^{kl}_{ij}$ is given by ${R^\sharp}^{kl}_{ij} = R^{ik}_{jl}$. Therefore, the invertibility of $R^\sharp$ can be expressed as follows: there exists a matrix P such that $P^{im}_{jn} R^{nk}_{ml} = \delta^i_l \delta^k_j$. Define the following algebras:

$$E_R := k\langle z^1_1, z^1_2, \ldots, z^d_d\rangle / \left(z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l\right),$$
$$H_R := k\langle z^1_1, z^1_2, \ldots, z^d_d, t^1_1,\ t^1_2, \ldots, t^d_d\rangle / \left(z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l,\ z^i_k t^k_j = t^i_k z^k_j = \delta^i_j\right),$$

where $\{z^i_j\}$ and $\{t^i_j\}$ are sets of generators. The algebra E_R is in fact a bialgebra with coproduct $\Delta(z^i_j) = z^i_k \otimes z^k_j$, $\varepsilon(z^i_j) = \delta^i_j$ and H_R is a Hopf algebra with $\Delta(z^i_j) = z^i_k \otimes z^k_j$, $\Delta(t^i_j) = z^k_j \otimes z^i_k$, $\varepsilon(z^i_j) = \varepsilon(t^i_j) = \delta^i_j$ and $S(z^i_j) = t^i_j$ [7, Theorem 2.1.1]. Moreover, the natural homomorphism of bialgebra $E_R \to H_R$ is injective [7, Theorem 2.3.5] thus E_R can be considered as a subbialgebra of H_R and simple E_R-comodules remain simple when considered as H_R-comodules.

The bialgebra E_R is considered as the function algebra on a quantum semigroup of type A and the Hopf algebra H_R is considered as the function algebra on a matrix quantum groups of A. The representations of this (semi-)group are thus comodules over H_R (respectively E_R).

1.2. Comodules over E_R

The space V is a comodule over E_R by the map $\delta : V \to V \otimes E_R;\ x_i \mapsto x_j \otimes z^j_i$. Since E_R is a bialgebra, any tensor power is also a comodule over E_R. The map $R : V \otimes V \to V \otimes V$ is a comodule map. The classification of E_R-comodules is done with the help of the action of the Hecke algebra.

1.2.1. The Hecke algebras and simple comodules

The Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ has generators T_i, $1 \leqslant i \leqslant n-1$, subject to the relations: $T_i T_j = T_j T_i$, $|i-j| \geqslant 2$; $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $1 \leqslant i \leqslant n-2$; $T_i^2 = (q-1)T_i + q$. There is a k-basis in $\mathcal{H}_n$ indexed by permutations of n elements: T_w, $w \in \mathfrak{S}_n$ ($\mathfrak{S}_n$ is the permutation group), in such a way that $T_{(i,i+1)} = T_i$ and $T_w T_v = T_{wv}$ if the length of wv is equal to the sum of the length of w and the length of v. If q is not a root of unity of degree greater than 1, $\mathcal{H}_n$ is a semisimple algebra. For more details, the reader is referred to [2,3].

The Hecke symmetry R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on $V^{\otimes n}$, $T_i \mapsto R_i = \mathrm{id}^{i-1} \otimes R \otimes \mathrm{id}^{n-i-1}$ which commutes with the coaction of E_R. The action of T_w will be denoted by R_w.

Thus, each element of $\mathcal{H}_n$ determines an endomorphism of $V^{\otimes n}$ as E_R-comodule. For q not a root of unity of degree greater 1, the converse is also true: each endomorphism of $V^{\otimes n}$ represents the action of an element of $\mathcal{H}_n$, moreover $V^{\otimes n}$ is semisimple and its simple subcomodules can be given as the images of the endomorphisms determined by primitive idempotents of $\mathcal{H}_n$, conjugate idempotents determine isomorphic comodules [6].

Since conjugate classes of primitive idempotents of $\mathcal{H}_n$ are indexed by partitions of n, simple subcomodules of $V^{\otimes n}$ are indexed by a subset of partitions of n. Thus E_R is cosemisimple and its simple comodules are indexed by a subset of partitions.

Example (*Quantum symmetrizers*). The primitive idempotent corresponding to partition (n) of n

$$X_n := \frac{1}{[n]_q} \sum_{w \in \mathfrak{S}_n} R_w$$

determines a simple comodule $\mathbf{S}_n$ called the nth quantum symmetric tensor power and the primitive idempotent corresponding to partition (1^n) of n

$$Y_n := \frac{1}{[n]_{1/q}} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} R_w$$

determines a simple comodule $\wedge_n$ called the nth quantum anti-symmetric tensor power.

1.2.2. The decomposition of the tensor products

The decomposition of the tensor product of two simple comodules can be given in terms of the Littlewood–Richardson coefficients.

Let I_λ denote the simple comodule corresponding to the partition λ. Then I_λ and I_μ can be realized as the images of two primitive idempotents $e_\lambda \in \mathcal{H}_r$ and $e_\mu \in \mathcal{H}_s$. Thus $I_\lambda \otimes I_\mu$ is the image of a (not necessarily primitive) idempotent in $\mathcal{H}_{r+s}$. This idempotent decomposes into an orthogonal sum of primitive idempotents, which yields a decomposition of I_λ and I_μ into direct sum of simple subcomodules. Taking into account that conjugate idempotents define isomorphic comodules, we have [6]

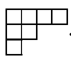
$$I_\lambda \otimes I_\mu \cong \bigoplus_\gamma I_\gamma^{\oplus c_{\lambda\mu}^\gamma} \tag{1.1}$$

where $c_{\lambda\mu}^\gamma$ is the Littlewood–Richardson coefficient describing the multiplicity of the Schur function s_γ in the product two other Schur functions s_λ and s_μ. The coefficients $c_{\lambda\mu}^\gamma$ can be computed by the following algorithm.

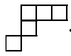
1.2.3. The Littlewood–Richardson algorithm

Let $[\lambda]$ be the diagram for a partition λ, $[\lambda] := \{(i, j); 1 \leqslant j \leqslant \lambda_i\}$. We usually represent $[\lambda]$ by a so-called Young diagram.

Example. If $\lambda = (4, 2, 1)$ then

$[\lambda] =$ .

The kth row (respectively, column) of the diagram consists of those nodes whose first (respectively, second) coordinate is k. We also make use of skew partitions. Let γ and λ be partitions with $\gamma_i \geqslant \lambda_i$ for all i. We define $[\gamma\backslash\lambda] := \{(i,j)\colon (i,j) \in [\gamma], \lambda_i < j \leqslant \gamma_i\}$. For example, in case $\gamma = (4,2,1)$, $\lambda = (1,1)$,

$[\gamma\backslash\lambda] =$  .

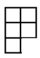
Let μ be a partition. A sequence of positive integers is said to have type μ if, for each i, i occurs μ_i times. Given a sequence of positive integers, each term is determined to be "good" or "bad" as follows:

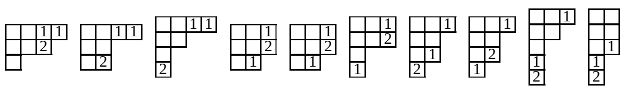
(i) all the 1's are good;
(ii) an $i+1$ is good if and only if the number of previous good i's is strictly greater than the number of previous good $(i+1)$'s.

A sequence is said to be good if all its terms are good. For example, the good sequences of type $\mu = (2,1)$ are 112, 121.

The algorithm for calculating $c^{\gamma}_{\lambda\mu}$, where λ is a partition of n, μ is a partition of m and γ is a partition of $m+n$, is described as follows:

(i) if $\lambda_i > \gamma_i$ for some i then $c^{\gamma}_{\lambda\mu} = 0$;
(ii) if $\lambda_i \leqslant \gamma_i$ for every i, then $c^{\gamma}_{\lambda\mu}$ is the number of ways of replacing the nodes of $[\gamma \setminus \lambda]$ by integers, such that
   - each k occurs μ_k times;
   - the numbers are non-decreasing along rows and strictly increasing down columns;
   - when reading from right to left in successive rows, we have a good sequence of type μ.

Example. Let $[\lambda] =$ , $[\mu] =$ . There are two good sequences of type $(2,1)$: (112), (121). We have the following possibilities of γ for which $c^{\gamma}_{\lambda\mu} \neq 0$:



which means

$$I_{(2,2,1)} \otimes I_{(2,1)} = I_{(4,3,1)} \oplus I_{(4,2^2)} \oplus I_{(4,2,1^2)} \oplus I_{(3^2,2)} \oplus I_{(3^2,1^2)} \oplus I^{\oplus 2}_{(3,2^2,1)} \oplus I_{(3,2,1^3)} \oplus I_{(2^3,1^2)}.$$

1.3. H_R-comodules

Since H_R is a Hopf algebra, for each finite-dimensional comodule M there is a comodule structure on $M^* = \mathrm{Hom}_k(M,k)$, for which the evaluation map $\mathrm{ev}\colon \phi \otimes v \mapsto \phi(v)$ is a morphism of comodules. The coaction on M^* is defined from the coaction on M as follows. Let (e_i) be a basis of M and (f^i) be its dual basis in M^*. Let the coaction ρ_M be given by $\rho_M(e_i) = e_j \otimes a_i^j$. Then the coaction ρ_{M^*} is given by

$$\rho_{M^*}: f^i \mapsto f^j \otimes S\big(a_j^i\big),$$

where S is the antipode. The linear map $\mathrm{db}: k \to M \otimes M^*$, $1 \mapsto \sum x_i \otimes \xi^i$ is obviously independent of the choice of the basis of M and can be checked to be a map of H_R-comodules. In the rest of this paper, the set of homomorphisms between two H_R comodules M, N will be denoted by $\mathrm{Hom}(M,N)$.

For any finite-dimensional H_R-comodule N, we have the following isomorphisms:

$$\mathrm{Hom}(M \otimes N, P) \overset{\varphi}{\cong} \mathrm{Hom}(M, P \otimes N^*): f \leftrightarrow g \tag{1.2}$$

given by $g = (f \otimes \mathrm{id}_{N^*})(\mathrm{id}_M \otimes \mathrm{db}_N)$, $f = (\mathrm{id}_P \otimes \mathrm{ev}_N)(g \otimes \mathrm{id}_N)$,

$$\mathrm{Hom}(M, N \otimes P) \overset{\phi}{\cong} \mathrm{Hom}(N^* \otimes M, P): h \leftrightarrow k. \tag{1.3}$$

given by $k = (\mathrm{ev}_N \otimes \mathrm{id}_P)(\mathrm{id}_{N^*} \otimes h)$, $h = (\mathrm{id}_N \otimes k)(\mathrm{db}_N \otimes \mathrm{id}_M)$.

Since R satisfies the Yang–Baxter equation, there exists a coquasitriangular structure on H_R inducing a braiding in the category of its right comodules, denoted by τ:

$$\tau_{M,N}: M \otimes N \to N \otimes M.$$

Notice that, since the coquasitriangular structure on E_R is the restriction of the one on H_R, the braiding for E_R-comodules is the same when considered as H_R-comodules. In particular, we have $\tau_{V,V} = R$, $\tau_{V,V^*} = P$, and $\tau_{V^*,V} = (R^{-1})^\sharp$.

Using the braiding, we deduce from (1.2) and (1.3) the following isomorphisms:

$$\mathrm{Hom}(N \otimes M, P) \cong \mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, P \otimes N^*) \cong \mathrm{Hom}(M, N^* \otimes P), \tag{1.4}$$

$$\mathrm{Hom}(M, P \otimes N) \cong \mathrm{Hom}(M, N \otimes P) \cong \mathrm{Hom}(N^* \otimes M, P) \cong \mathrm{Hom}(M \otimes N^*, P). \tag{1.5}$$

1.4. The birank and dimension of comodules

Consider the series

$$P_{\mathsf{S}}(t) := \sum_{i=0}^{\infty} \dim \mathsf{S}_i \, t^i.$$

Using formula (1.1), we deduce [6] that $P_{\mathsf{S}}(t)$ is a rational function and can be given as the quotient of a polynomial with all roots real and negative by a polynomial with all roots real and positive:

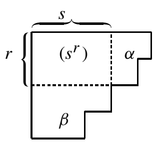
$$P_{\mathsf{S}}(t) = \frac{\prod_{i=1}^{s}(1 + x_i t)}{\prod_{j=1}^{r}(1 - y_j t)}, \quad x_i, y_j > 0.$$

Moreover, both polynomials are reciprocal with integral coefficients [4]. The pair (r, s) of degrees of these polynomials is called the *birank* of R.

Using the birank, we can characterize those partitions which determine simple comodules of E_R: *the comodule I_λ is non-zero and hence simple if and only if λ satisfies $\lambda_{r+1} \leqslant s$, the set of such partitions is denoted by $\Gamma_{r,s}$* [6, Theorem 3.5].

1.4.1. Dimension

In principle, the k-dimension of the simple comodule I_λ, $\lambda \in \Gamma_{r,s}$, can be computed in terms of the coefficients of the Poincaré series of S, although the formula is quite complicated. However, for partitions λ satisfying the condition $\lambda_r \geqslant s$, there is a simpler formula. Namely, for such a partition we have the decomposition $\lambda = (s^r) + \alpha \cup \beta$, where α has at most r non-zero components and β has $\beta_1 \leqslant s$:



and

$$\dim I_\lambda = \prod_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} (x_i + y_j) \cdot s_\alpha(x) \cdot s_{\beta'}(y) \tag{1.6}$$

where $s_\alpha(x)$ (respectively $s_\beta(y)$) is the Schur function on the parameters $x_1, x_2, \ldots, x_r$ (respectively $y_1, y_2, \ldots, y_s$), β' is the partition conjugate to β [11, Chapter I] (this goes back to a formula of Berele and Regev [1]).

1.5. The Koszul complex

The Koszul complex associated to R has the (k, l) term $K_{k,l} := \wedge_k \otimes \mathsf{S}_l^* = I_{1,1,k-2} \otimes I_{l,0,0}^*$, the differential $d_{k,l}$ is given by

$$d_{k,l} : \wedge_k \otimes \mathsf{S}_l^* \to V^{\otimes l} \otimes V^{*\otimes l} \xrightarrow{\mathrm{id} \otimes \mathrm{db}_V \otimes \mathrm{id}} V^{\otimes k+1} \otimes V^{*\otimes l+1} \xrightarrow{Y_{k+1} \otimes X_{l+1}^*} \wedge_{k+1} \otimes \mathsf{S}_{l+1}^*,$$

where X_l, Y_k are the q-symmetrizer operators introduced in Section 1.2. The reader is referred to [5] for the proof that d is a differential.

Notice that our complex is in fact a collection of diagonal subcomplexes. Denote K_a the subcomplex consisting of the terms $K_{k,l}$ with $k-l=a$. Define another differential $\partial_{k,l}$ as follows:

$$\partial_{k,l}: \wedge_{k+1} \otimes \mathsf{S}^*_{l+1} \to V^{\otimes k+1} \otimes V^{*\otimes l+1} \xrightarrow{\mathrm{id}\otimes \mathrm{ev}_V\, \tau_{V\otimes V^*}\otimes \mathrm{id}} V^{\otimes k} \otimes V^{*\otimes l} \xrightarrow{Y_k \otimes X^*_l} \wedge_k \otimes \mathsf{S}^*_l .$$

Let $\mathrm{rank}_q R := \sum_{i,j} P^{ij}_{ij}$, the *quantum rank of R*, where P is the inverse to the half adjoint of R. Then d and ∂ satisfy [5]:

$$(qd\partial + \partial d)|_{K_{k,l}} = q^k([l-k]_q + \mathrm{rank}_q R). \tag{1.7}$$

If $-[l-k]_q \neq \mathrm{rank}_q R$, then the homology group at the term (k,l) vanishes. Moreover, we have

$$K_{k,l} = \wedge_k \otimes \mathsf{S}^*_l \cong \mathrm{Im}\, d_{k-1,l-1} \oplus \mathrm{Im}\, \partial_{k,l} \tag{1.8}$$

and therefore $\mathrm{Im}\,\partial_{k,l} \cong \mathrm{Im}\, d_{k,l}$. Indeed, according to (1.7), we have, for $x \in K_{k,l}$:

$$qd_{k-1,l-1}\partial_{k-1,l-1}(x) + \partial_{k,l}d_{k,l}(x) = q^k(\mathrm{rank}_q R + [l-k]_q)x,$$

thus $x \in \mathrm{Im}\,\partial_{k,l} + \mathrm{Im}\, d_{k-1,l-1}$. Consider $x \in \mathrm{Im}\,\partial_{k,l} \cap \mathrm{Im}\, d_{k-1,l-1}$, which means that $x = d_{k-1,l-1}\partial_{k-1,l-1}(y)$, $x = \partial_{k,l}d_{k,l}(z)$. Then $d_{k,l}(x) = 0$ and $\partial_{k-1,l-1}(x) = 0$. According to (1.7), $x = 0$. Thus $\mathrm{Im}\,\partial_{k,l} \cap \mathrm{Im}\, d_{k,l} = 0$, hence $K_{k,l} \cong \mathrm{Im}\, d_{k-1,l-1} \oplus \mathrm{Im}\,\partial_{k,l}$.

Lemma 1.1. *Let R be a Hecke symmetry with birank (r,s), $r,s \geqslant 1$. Then the differentials $d_{k,l}$ are non-zero for all pairs (k,l), $k,l \geqslant 0$.*

Proof. Remember that S_l and $\wedge_k$ are direct summands of $V^{\otimes l}$ and $V^{\otimes k}$, given by means of the operators $X_l = \rho(x_l)$ and $Y = \rho(y_k)$, respectively. On the other hand, we have $X_{l+1}(X_l \otimes \mathrm{id}_V) = X_{l+1}$, $Y_{k+1}(\mathrm{id}_V \otimes Y_k) = Y_{k+1}$. Hence, to show $d_{k,l} \neq 0$ is equivalent to showing that the map

$$D := \left(Y_{k+1} \otimes X^*_{l+1}\right)\left(\mathrm{id}_V^{\otimes k} \otimes \mathrm{db} \otimes \mathrm{id}_{V^*}^{\otimes l+1}\right)$$

which is an element of $\mathrm{Hom}(V^{\otimes k} \otimes V^{*\otimes l}, V^{\otimes k+1} \otimes V^{*\otimes l+1})$, is non-zero. According to (1.4) and (1.5), we have the isomorphism

$$\mathrm{Hom}\left(V^{\otimes k} \otimes V^{*\otimes l}, V^{\otimes k+1} \otimes V^{*\otimes l+1}\right) \cong \mathrm{Hom}\left(V^{\otimes k} \otimes V^{\otimes l+1}, V^{\otimes l} \otimes V^{\otimes k+1}\right)$$

under which, D is mapped to the map

$$\begin{aligned} g &= \left(\mathrm{id}_V^{\otimes l} \otimes Y_{k+1}\right)(\tau_{V^{\otimes k}, V^{\otimes l}} \otimes \mathrm{id}_V)\left(\mathrm{id}_V^{\otimes k} \otimes X_{k+1}\right) \\ &= q^{-l}\tau_{V^{\otimes l}, V^{\otimes k+1}}\left(Y_{k+1} \otimes \mathrm{id}_V^{\otimes k+1}\right)\left(\mathrm{id}_V^{\otimes k} \otimes X_{l+1}\right). \end{aligned}$$

According to [2, Section 1], the element $y_{k+1}T_w x_{l+1}$, where w is the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k+1 & k+2 & \cdots & l+k+1 \\ 1 & l+2 & \cdots & l+k+1 & 2 & \cdots & l+1 \end{pmatrix},$$

generates the (left) Specht module (minimal left ideal of $\mathcal{H}_n$) corresponding to partition $\lambda = (l+1, 1^k)$. On the other hand, according to the discussion in Section 1.4, $I_\lambda \neq 0$ for $\lambda \in \Gamma_{r,s}$ and the primitive idempotent corresponding to this partition defines a non-zero endomorphism. Consequently, $\rho(y_{k+1}T_w x_{l+1}) \neq 0$ (this element is not an idempotent but each Specht module contains a primitive idempotent which is its multiple). Thus

$$\rho(y_{k+1}T_w x_{l+1}) = 0 \neq \left(Y_{k+1} \otimes \mathrm{id}_V^{\otimes l}\right) R_w \left(X_{l+1} \otimes \mathrm{id}_V^{\otimes k}\right) = \left(Y_{k+1} \otimes \mathrm{id}_V^{\otimes l}\right)\left(\mathrm{id}_V^{\otimes k} \otimes X_{l+1}\right) R_w.$$

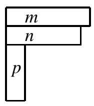
Hence $g \neq 0$, implying $d_{k,l} \neq 0$. □

2. Quantum linear groups of type (1, 0)

We assume from now on that R has birank $(2,1)$, which means that the series $P_{\mathsf{S}}(t)$ has the form

$$P_{\mathsf{S}}(t) = \frac{1+t}{(1-ut)(1-u^{-1}t)},$$

where $u \in R^+$, $u + u^{-1} \in \mathbb{Z}$. If $\lambda \in \Gamma_{21}$, then λ has the form $(m, n, 1^p)$ with



$$m \geqslant n \geqslant 0, \qquad p \geqslant 0 \quad \text{and} \quad \text{if } n = 0 \text{ then } p = 0. \tag{2.1}$$

A triple of integers satisfying the above condition is said to be *corresponding to a partition*. For such a triple, let $\lambda = (m, n, 1^p)$ and set

$$I_{m,n,p} := I_\lambda.$$

Using Eq. (1.1) and the Littlewood–Richardson algorithm, we can decompose the tensor products of these comodules. Below are some formulas which will be frequently used:

$$I_{m,n,p} \otimes I_{1,0,0} = \begin{cases} I_{m+1,n,p} + I_{m,n+1,p} + I_{m,n,p+1} & \text{if } m > n, \\ I_{m+1,n,p} + I_{m,n,p+1} & \text{if } m = n, \end{cases} \tag{2.2}$$

$$I_{m,m,1} \otimes I_{n,0,0} = I_{m+n,m,1} + I_{m+n-1,m,2}, \quad m, n \geqslant 1, \tag{2.3}$$

$$I_{m,n,p} \otimes I_{1,1,k} = \begin{cases} I_{m+1,n+1,p+k} + I_{m+1,n,p+k+1} + I_{m,n+1,p+k+1} + I_{m,n,p+k+2} \\ \hfill \text{if } m > n, \\ I_{m+1,m+1,p+k} + I_{m+1,m,p+k+1} + I_{m,m,p+k+2} \quad \text{if } m = n. \end{cases} \tag{2.4}$$

We first compute the dimension of these comodules in terms of the series $P_S(t)$. Denote

$$(m)_u := \frac{u^m - u^{-m}}{u - u^{-1}},$$

which are integers for all $m \in \mathbb{Z}$. Notice that, since u is real positive, $(m)_u = (n)_n$ if and only if $m = n$.

It follows immediately from the formula of $P_{\mathsf{S}}(t)$ that $\dim I_{m,0,0} = \dim \mathsf{S}_n = (m)_u + (m+1)_u$. For $n \geqslant 1$ we have, according to (1.6),

$$\dim I_{m,n,p} = \big((2)_u + 2\big) S_{(m-1,n-1)}\big(u, u^{-1}\big) S_{(p)}(1) = \big((2)_u + 2\big)(m-n+1)_u. \tag{2.5}$$

2.1. Decomposition of tensor products with dual comodules

We mention some decomposition rules of tensor products of H_R-comodules which follow immediately from Eqs. (2.2), (2.3). For each triple (m, n, p) corresponding to a partition, the E_R-comodule $I_{m,n,p}$ is also a simple comodule over H_R (see Section 1.1).

Lemma 2.1. *For a triple (m, n, p) with $m \geqslant n \geqslant 2$, $p \geqslant 1$,*

$$I_{m,n,p} \otimes I^*_{1,0,0} = \begin{cases} I_{m-1,n,p} + I_{m,n-1,p} + I_{m,n,p-1} & \text{if } m > n, \\ I_{m,n-1,p} + I_{m,n,p-1} & \text{if } m = n. \end{cases} \tag{2.6}$$

Proof. Using the isomorphisms (1.2) and the formula (2.2), we have

$$\operatorname{Hom}(I_{m,n,p} \otimes I^*_{1,0,0}, I_{m,n-1,p}) \cong \operatorname{Hom}(I_{m,n,p}, I_{m,n-1,p} \otimes I_{1,0,0}) = k.$$

Therefore $I_{m,n,p} \otimes I^*_{1,0,0}$ contains one copy of $I_{m,n-1,p}$. Similarly, it contain one copy of each comodule on the right-hand side. Counting the dimension of the comodules, we conclude that it does not contain any other comodules (recall that the comodules on the right-hand side of (2.6) are simple). □

Using a similar argument, we deduce from (2.3):

$$I_{m,m,1} \otimes I^*_{n,0,0} = I_{m,m-n,1} + I_{m,m-n+1,0}, \quad m > n \geqslant 1. \tag{2.7}$$

Since E_R is a subbialgebra of H_R, its simple comodules are also simple over H_R. Unfortunately, H_R is not cosemisimple. The aim of this paper is to classify simple comodules of H_R. Our main tool is the Koszul complex, which in this case has the following form:

$$K_1 : 0 \to I_{1,0,0} \xrightarrow{d_{1,0}} I_{1,1,0} \otimes I^*_{1,0,0} \cdots I_{1,1,k-2} \otimes I^*_{k-1,0,0} \xrightarrow{d_{k,k-1}} I_{1,1,k-1} \otimes I^*_{k,0,0} \cdots,$$

$$K_a\ (a \geqslant 2) : 0 \to I_{1,1,a-2} \xrightarrow{d_{a,0}} I_{1,1,a-1} \otimes I^*_{1,0,0} \xrightarrow{d_{a+1,1}} I_{1,1,a} \otimes I^*_{2,0,0} \to \cdots,$$

$$K_a\ (a \leqslant 0) : 0 \to I^*_{-a,0,0} \xrightarrow{d_{0,a}} I_{1,0,0} \otimes I^*_{1-a,0,0} \xrightarrow{d_{1,1+a}} I_{1,1,0} \otimes I^*_{2-a,0,0} \to \cdots.$$

Lemma 2.2. *The Koszul complex K_1 has a non-zero homology at the term $(2, 1)$.*

Proof. Consider the complex K_1. Consider the tensor product of $I_{3,3,1}$ with all terms of K_1. We have $I_{3,3,1} \otimes I_{1,0,0} = I_{4,3,1} + I_{3,3,2}$ and, according to (2.6), (2.7),

$$\begin{aligned} I_{3,3,1} \otimes I_{1,1,0} \otimes I^*_{1,0,0} &= I_{1,1,0} \otimes (I_{3,3,0} + I_{3,2,1}) \\ &= I_{4,4,0} + 2I_{4,3,1} + 2I_{3,3,2} + I_{4,2,2} + I_{3,2,3}, \\ I_{3,3,1} \otimes I_{1,1,1} \otimes I^*_{2,0,0} &= I_{1,1,1} \otimes (I_{3,1,1} + I_{3,2,0}) \\ &= 2I_{4,2,2} + I_{4,1,3} + 2I_{3,2,3} + I_{3,1,4} + I_{4,3,1} + I_{3,3,2}. \end{aligned}$$

We see that $I_{4,4,0}$ is a subcomodule of $I_{3,3,1} \otimes I_{1,1,0} \otimes I^*_{1,0,0}$ and $I_{4,4,0}$ is not a subcomodule of $I_{3,3,1} \otimes I_{1,1,1} \otimes I^*_{2,0,0}$, therefore $I_{4,4,0}$ is a subcomodule of $I_{3,3,1} \otimes \operatorname{Ker} d_{2,1}$. Thus $\operatorname{Ker} d_{2,1} \neq \operatorname{Im} d_{1,0}$, which proves the assertion. □

2.2. The integral and splitting comodules

A right integral on a Hopf algebra H is an H-comodule homomorphism $H \to k$, where H coacts (from the right) on itself by the coproduct and on k by the unit map. Left integral is defined in the similar manner.

An immediate consequence of the above lemma is that the quantum rank of a Hecke symmetry of birank $(2, 1)$ is $-[-1]_q = q^{-1}$. Hence, according to [9, Theorem 3.2], there exists a left integral on H_R that remains unchanged when composed with the antipode; whence it is also a right integral. In other words, left and right integrals on H_R coincide.

A simple comodule over H_R is called *splitting* if it is projective and injective (thus it splits in any comodule). The following lemma follows immediately from [8, Theorem 3.1] and [9, Proposition 5.1].

Lemma 2.3. *Let R be a Hecke symmetry of birank* $(2, 1)$. *Then for any partition $\lambda = (m, n, 1^p) \in \Gamma_{2,1}$, the corresponding simple comodule I_λ is splitting if and only if $n \geqslant 1$. In particular, for all $n \geqslant 2$, $\wedge_n = I_{1,1,n-2}$ are splitting; on the other hand, $\mathsf{S}_n = I_{n,0,0}$ is not splitting for all n and the field $k =: I_{0,0,0}$ itseft is not splitting.*

The following lemma is useful for knowing which comodule is splitting.

Lemma 2.4. *Let H_R be a coquasitriangular Hopf algebra on which a left integral exists and is also a right integral. Let M be a projective, injective comodule with* $\operatorname{End}(M) \cong k$. *Then M is splitting.*

Proof. (See [8] for notations.) Let S be the socle of M (i.e., the sum of all its simple subcomodules). Since M is indecomposable, it is the injective envelope of S and S is simple. By [8, Theorem 2.3], $\operatorname{Hom}(M, S^\bullet) \neq 0$. On the other hand, since left and right integrals on H_R coincide, the distinguished element in H_R is 1, and since H_R is coquasitriangular, $S \cong S^{**}$. Therefore $S \cong S^\bullet$. Thus $\operatorname{Hom}(M, S) \neq 0$. Were $S \neq M$, we would have $\dim \operatorname{End}(M) \geqslant 2$, a contradiction; therefore $S = M$. Thus, M is splitting. □

Corollary 2.5. *The comodules* $\operatorname{Im} d_{k,l}$ *are simple comodule for all pairs (k, l) with $l, k \geqslant 0$, $k - l \neq 1$.*

Proof. For $k=0$, $\operatorname{Im} d_{0,l} = \mathsf{S}_k^*$ is simple. Let $k \geqslant 1$. We have, according to (1.4), (1.5), $\dim \operatorname{End}(\wedge_{k+1} \otimes \mathsf{S}_{l+1}^*) = \dim \operatorname{End}(\wedge_{k+1} \otimes \mathsf{S}_{l+1}) = 2$. On the other hand, since $\wedge_{k+1}$ for $k \geqslant 1$ is splitting (thus injective and projective), $\wedge_{k+1} \otimes \mathsf{S}_{l+1}^*$ is projective and injective, hence so is $\operatorname{Im} d_{k,l}$, being direct a summand for $k-l \neq 1$ (cf. (1.8)). Therefore $\dim \operatorname{End}(\operatorname{Im} d_{k-1,l-1}) = \dim \operatorname{End}(\operatorname{Im} \partial_{k,l}) = 1$. According to Lemma 2.4, $\operatorname{Im} d_{k,l}$ is simple. □

Let us denote for any $l, k \geqslant 0$:

$$I_{1,-l,k} := \begin{cases} \operatorname{Im} d_{k+1,l} & \text{if } l > k \geqslant 0, \\ \operatorname{Im} d_{k+2,l+1} & \text{if } k > l \geqslant 0. \end{cases} \tag{2.8}$$

According to (1.8), the following equalities hold:

$$\begin{aligned} I_{1,0,0} \otimes I_{l,0,0}^* &= I_{l-1,0,0}^* + I_{1,-l,0}, \quad l > k = 0, \\ I_{1,1,k-1} \otimes I_{l,0,0}^* &= I_{1,1-l,k-1} \oplus I_{1,-l,k}, \quad l > k \geqslant 1, \\ I_{1,1,k} \otimes I_{1,0,0}^* &= *I_{1,1,k-1} \oplus I_{1,0,k}, \quad k \geqslant 1, \\ I_{1,1,k} \otimes I_{l+1,0,0}^* &= I_{1,-l,k} \oplus I_{1,1-l,k-1}, \quad k > l \geqslant 1. \end{aligned} \tag{2.9}$$

and according to Corollary 2.5, $I_{1,-l,k}$ are splitting for all $k \neq l \geqslant 0$.

For all $l > k \geqslant 1$, we have

$$\dim I_{1,-l,k} = \big((2)_u + 2\big)(l+1)_u. \tag{2.10}$$

Indeed, for $k=1$ we have, according to (2.9),

$$\dim I_{1,-l,1} = \dim I_{1,0,0} \cdot \dim I_{0,-l,0} - \dim I_{0,-l+1,0} = \big((2)_u + 2\big)(l+1)_u.$$

For $k \geqslant 2$, using induction, we have

$$\begin{aligned} \dim I_{1,-l,k} &= \dim I_{1,1,k-1} \cdot \dim I_{0,-l,0} - \dim I_{1,-l+1,k-1} \\ &= \big((2)_u + 2\big)\big((l)_u + (l+1)_u\big) - (A+2)(l)_u \\ &= \big((2)_u + 2\big)(l+1)_u. \end{aligned}$$

Similarly, we have, for $k > l \geqslant 1$,

$$\dim I_{1,-l,k} = \big((2)_u + 2\big)(l+2)_u. \tag{2.11}$$

3. The homology of the Koszul complex K_1

In this section, we show that the Koszul complex K_1 is exact at every term except at the term $(2,1)$, where it has the homology of dimension 1 over k. We will compute the tensor product of the homology with known simple comodules, these formula play a crucial role in classifying simple comodules of H_R.

The Koszul complex K_1 has the form (with $d_k := d_{k,k-1}$)

$$0 \longrightarrow I_{1,0,0} \xrightarrow{d_1} I_{1,1,0} \otimes I^*_{1,0,0} \xrightarrow{d_2} I_{1,1,1} \otimes I^*_{2,0,0} \xrightarrow{d_3} I_{1,1,2} \otimes I^*_{3,0,0} \longrightarrow \cdots.$$

Recall that we also have the differentials $\partial_k = \partial_{k,k-1}$ forming a complex

$$0 \longleftarrow I_{1,0,0} \xleftarrow{\partial_1} I_{1,1,0} \otimes I^*_{1,0,0} \xleftarrow{\partial_2} I_{1,1,1} \otimes I^*_{2,0,0} \xleftarrow{\partial_3} I_{1,1,2} \otimes I^*_{3,0,0} \longleftarrow \cdots.$$

Theorem 3.1. *Let R be a Hecke symmetry of birank $(2,1)$, then the homology of the associated complex K_1 at the term $(2,1)$ has dimension one over k. Denote this comodule by $I_{1,1,-1}$. Then it is invertible, i.e. $I_{1,1,-1} \otimes I^*_{1,1,-1} = I^*_{1,1,-1} \otimes I_{1,1,-1} = k$ and for any simple comodule $I_{m,n,p}$ with $m \geqslant n \geqslant 1$, $p \geqslant 1$,*

$$I_{m,n,p} \otimes I_{1,1,-1} = I_{m+1,n+1,p-1}. \tag{3.1}$$

Proof. Since $d_1 \neq 0$, it is injective, for $I_{1,0,0}$ is simple. Analogously, since $\partial_1 \neq 0$, it is surjective. Since $\operatorname{Im} d_1\partial_1 = I_{1,0,0}$, $d_1\partial_1 \neq 0$. On the other hand, according to Lemma 2.2, K_1 is not exact, hence $q\partial_1 d_2 + d_1\partial_1 = 0$, since for d_1, $\operatorname{Ker} d_2 \subset \operatorname{Ker} \partial_1$. Thus, we have the following inclusions:

$$0 \subseteq I_{1,0,0} \cong \operatorname{Im} d_1 \subseteq \operatorname{Ker} d_2 \subseteq \operatorname{Ker} \partial_1 \subseteq I_{1,1,0} \otimes I^*_{1,0,0} =: X. \tag{3.2}$$

We show that comodules in this series are distinct. Since $X/\operatorname{Ker}\partial_1 = I_{1,0,0}$ and since $I_{1,0,0}$ is not a subcomodule of $I_{1,1,1} \otimes I^*_{2,0,0}$, $\operatorname{Ker} d_2$ is a strict subcomodule of $\operatorname{Ker}\partial_1$. Since ∂_1 is surjective, we have $X/\operatorname{Ker}\partial_1 \cong \operatorname{Im}\partial_1 = I_{1,0,0}$, therefore $X \neq \operatorname{Ker}\partial_1$. Finally, $\operatorname{Im} d_1 \neq \operatorname{Ker} d_2$, according to Lemma 2.2.

Let $I_{1,1,-1} := \operatorname{Ker} d_2/\operatorname{Im} d_1$, we show that this comodule has dimension 1. According to Lemma 2.2, we have

$$I_{3,3,1} \otimes (\operatorname{Ker}\partial_1/\operatorname{Im} d_1) \cong I_{4,2,2} + I_{3,2,3} + I_{4,4,0} \tag{3.3}$$

and $I_{1,1,-1} \otimes I_{3,3,1}$ contains a comodule isomorphic to $I_{4,4,0}$. Assume $\dim I_{1,1,-1} \geqslant 2$. Then $(\operatorname{Ker} d_2/\operatorname{Im} d_1) \otimes I_{3,3,1}$ should contain comodules other than $I_{4,4,0}$. Thus

$$I_{3,3,1} \otimes (\operatorname{Ker}\partial_1/\operatorname{Ker} d_2) = \begin{cases} I_{3,2,3} \\ I_{4,2,2} \end{cases} \overset{(2.5)}{\Longrightarrow} \quad \dim(\operatorname{Ker}\partial_1/\operatorname{Ker} d_2) = \begin{cases} (2)_u \\ (3)_u \end{cases}. \tag{3.4}$$

Consider now the tensor product of K_1 with $I^*_{1,0,0}$. We have

$$I_{1,1,0} \otimes I^*_{1,0,0} \otimes I^*_{1,0,0} = I_{1,1,0} \otimes (I_{1,1,0} + I_{2,0,0})^*.$$

According to (2.9), $I^*_{2,0,0} \otimes I_{1,1,0} = I_{1,-1,0} + I_{1,-2,1}$, where $I_{1,-2,1}$ is simple with dimension $((2)_u + 2)(3)_u$. Therefore $I_{1,-2,1}$ should be isomorphic to a subquotient either of $I^*_{1,0,0} \otimes (\operatorname{Ker}\partial_1/\operatorname{Ker} d_2)$, or of $I^*_{1,0,0} \otimes (\operatorname{Ker} d_2/\operatorname{Im} d_1)$.

If $I_{1,-2,1}$ isomorphic a subquotient of $I_{0,-1,1} \otimes (\operatorname{Ker} \partial_1 / \operatorname{Ker} d_2)$, then

$$\dim I_{1,-2,1} \leqslant \max\bigl\{\bigl((2)_u + 1\bigr)(2)_u, \bigl((2)_u + 1\bigr)(3)_u\bigr\} < \bigl((2)_u + 2\bigr)(3)_u,$$

contradiction.

If $I_{1,-2,1}$ isomorphic a subquotient of $I^*_{1,0,0} \otimes (\operatorname{Ker} d_2 / \operatorname{Im} d_1)$, then

$$\dim I_{1,-1,2} \leqslant \max\bigl\{\bigl((2)_u + 1\bigr)^2, \bigl((2)_u + 1\bigr)(2)_u^2\bigr\}. \tag{3.5}$$

It is easy to check that (3.5) cannot happen if $(2)_u \geqslant 3$. There remains only one possibility $(2)_u = 2$, thus $u = 1$. Then we have $\dim I_{1,-2,1} = 12$, therefore $\dim I_{1,1,-1} = 4$, $\dim \operatorname{Ker} \partial_1 / \operatorname{Ker} d_2 = 2$, and $I_{1,1,-1} \otimes I^*_{1,0,0} \cong I_{1,-2,1}$ which is simple. According to (2.6) and (2.9), we have

$$I_{1,0,0} \otimes I_{1,1,0} \otimes I^*_{1,0,0} = (I_{1,1,1} + I_{2,1,0}) \otimes I^*_{1,0,0} = I_{1,1,0} + I_{1,0,1} + I_{2,1,0} \otimes I^*_{1,0,0}. \tag{3.6}$$

We have $\dim I_{1,0,1} = 8$, $\dim I_{1,0,0} \otimes (\operatorname{Ker} \partial_1 / \operatorname{Im} d_1) = 6$. Hence $I_{1,0,1}$ should be a subquotient of $I_{1,0,0} \otimes I_{1,1,-1}$. Since $I_{1,0,1}$ is splitting, $I_{1,1,-1} \otimes I_{1,0,0} = I_{1,0,1} \oplus N$, $N \neq 0$. Hence $\dim \operatorname{End}(I_{1,1,-1} \otimes I_{1,0,0}) \geqslant 2$, contradicting with the fact that

$$\dim \operatorname{End}(I_{1,1,-1} \otimes I_{1,0,0}) = \dim \operatorname{End}\bigl(I_{1,1,-1} \otimes I^*_{1,0,0}\bigr) = \dim \operatorname{End}(I_{1,-2,1}) = 1.$$

Thus both cases in (3.4) cannot happen, therefore $\dim I_{1,1,-1} = 1$.

Now, taking the tensor product of $I_{m,n,p}$, $m \geqslant n \geqslant 1$, $p \geqslant 1$ with K_1. As in Lemma 2.2, we can show that $I_{m+1,n+1,p-1} \subset \operatorname{Ker}(d_3 \otimes \mathrm{id}_{I_{m,n,p}})$ and $I_{m+1,n+1,p-1} \not\subseteq \operatorname{Im}(d_1 \otimes \mathrm{id})$, but $\dim I_{1,1,-1} = 1$, thus $I_{m,n,p} \otimes I_{1,1,-1} \cong I_{m+1,n+1,p-1}$. □

Corollary 3.2. *The quotient* $\operatorname{Ker} \partial_1 / \operatorname{Ker} d_2$ *is isomorphic to* $I_{1,-1,1} := I_{2,0,0} \otimes I^*_{1,1,-1}$. *Consequently, the composition series of* $K_{2,1} = I_{1,1,0} \otimes I_{1,0,0}$ *consists of* $I_{1,1,-1}$, $I_{1,-1,1}$ *and two copies of* $I_{1,0,0}$.

Proof. Let $Y := \operatorname{Ker} \partial_1 / \operatorname{Ker} d_2$. According to (3.2), this is a subcomodule of $X / \operatorname{Ker} d_2 = \operatorname{Im} d_2$ and $\operatorname{Im} d_2 / Y = I_{1,0,0}$. Moreover, since $\operatorname{Hom}(I_{1,0,0}, \wedge_3 \otimes \mathsf{S}_2^*) = 0$, $I_{1,0,0}$ cannot be a subcomodule of $\operatorname{Im} d_2$ (which is a subcomodule of $I_{1,1,1} \otimes I^*_{2,0,0}$).

On the other hand, we have $\operatorname{Hom}(I_{1,1,1} \otimes I^*_{2,0,0}, I_{1,-1,1})$ and $\operatorname{Hom}(I_{1,-1,1}, I_{1,1,1} \otimes I^*_{2,0,0})$ are both non-zero, that is $I_{1,-1,1}$ is at the same time a subcomodule and a quotient comodule of $I_{1,1,1} \otimes I^*_{2,0,0}$. However $I_{1,-1,1}$ is not injective, while $I_{1,1,1} \otimes I^*_{2,0,0}$ is, consequently the latter should contain two copies of the former in its composition series. We have:

$$I_{1,-1,1} \otimes I_{3,3,1} = I_{2,0,0} \otimes I^*_{1,1,-1} \otimes I_{3,3,1} = I_{3,2,3} + I_{4,2,2}, \tag{3.7}$$

$$Y \otimes I_{3,3,1} = I_{3,2,3} + I_{4,2,2},$$

$$\begin{aligned} I_{1,1,1} \otimes I^*_{2,0,0} \otimes I_{3,3,1} &= (I_{4,4,2} + I_{4,3,3} + I_{3,3,4}) \otimes I^*_{2,0,0} \\ &= I_{4,3,1} + 2 \cdot I_{4,2,2} + 2 \cdot I_{3,2,3} + I_{3,3,2} + I_{4,1,3} + I_{3,1,4}. \end{aligned} \tag{3.8}$$

Were $I_{1,-1,1} \neq Y$, Eq. (3.7) would imply that $I_{1,1,1} \otimes I^*_{2,0,0}$ contains 3 copies of $I_{4,2,2}$, contradicting with (3.8). □

Theorem 3.3. *Let R be a Hecke symmetry of birank $(2, 1)$. Then the homology of the Koszul complex at all terms $K_{k+1,k}$, $k \geqslant 2$, vanishes. Moreover, the composition series of $K_{k+1,k}$ consists of $I_{1,2-k,k-2}$, $I_{1,-k,k}$ and two copies of $I_{1,1-k,k-1}$.*

Proof. We proof the theorem for $k = 2$. Using the equation $d_2\partial_2 + \partial_3 d_3 = 0$, we have the following inclusions:

$$0 \neq \operatorname{Im} d_2 \subseteq \operatorname{Ker} d_3 \subseteq \operatorname{Ker} d_2.\partial_2 \subseteq I_{1,1,1} \otimes I^*_{2,0,0}. \tag{3.9}$$

We first show that $d_2\partial_2 \neq 0$. Assume the contrary, $d_2\partial_2 = 0$, then $\operatorname{Im}\partial_2 \subseteq \operatorname{Ker} d_2$. On the other hand, we have $\operatorname{Im}\partial_2 d_2 = \operatorname{Im} d_1\partial_1 = I_{1,0,0}$, therefore $\operatorname{Im}\partial_2 \supseteq I_{1,0,0}$. Since $\operatorname{Hom}(I_{1,1,1} \otimes I_{2,0,0}, I_{1,0,0}) = 0$, $\operatorname{Im}\partial_2 \neq I_{1,0,0}$. Since $\operatorname{Ker} d_2/I_{1,0,0} = I_{1,1,-1}$, $\operatorname{Im}\partial_2 = \operatorname{Ker} d_2$, which means $\operatorname{Im}\partial_2$ contains $I_{1,1,-1}$ in its composition series. This however leads to contradiction, for $\operatorname{Im}\partial_2$ is a quotient of $I_{1,1,1} \otimes I^*_{2,0,0}$ and the later comodule does not contain $I_{1,1,-1}$ in its composition series, a fact which implies from (3.7) and the equation $I_{3,3,1} \otimes I_{1,1,-1} = I_{4,4,0}$. Thus we have $d_2\partial_2 \neq 0$.

Further we show that $\operatorname{Im} d_2\partial_2 = I_{1,-1,1}$. Indeed, we have $\operatorname{Im} d_2\partial_2 \subseteq \operatorname{Im} d_2$, but looking at the chain (3.9) and the decompositions in (3.7) and (3.8) we see that $\operatorname{Im} d_2\partial_2 \neq \operatorname{Im} d_2$ – otherwise $I_{1,1,1} \otimes I^*_{2,0,0}$ would contain two copies of $I_{4,3,1}$ and of $I_{3,3,2}$. Thus $\operatorname{Im} d_2\partial_2$ is a non-zero, strict subcomodule of $\operatorname{Im} d_2$; hence equal to $I_{1,-1,1}$.

According to (3.9), $I_{1,-1,1} = \operatorname{Im} d_2\partial_2$ is a quotient comodule of $\operatorname{Im} d_3$. Since the former is not a subcomodule of $K_{4,3} = I_{1,1,2} \otimes I^*_{3,0,0}$ while the latter is, they are not isomorphic. Consequently $\operatorname{Ker} d_3 \neq \operatorname{Ker} d_2\partial_2$.

We have, according to (3.7), (3.8),

$$(\operatorname{Ker} d_3/\operatorname{Im} d_2) \otimes I_{3,3,1} + (\operatorname{Ker} d_2\partial_2/\operatorname{Ker} d_3) \otimes I_{3,3,1} \cong I_{4,1,3} + I_{3,1,4}.$$

Thus, assuming that $\operatorname{Im} d_2 \neq \operatorname{Ker} d_3$, we have

$$I_{3,3,1} \otimes (\operatorname{Ker} d_2\partial_2/\operatorname{Ker} d_3) = \begin{cases} I_{4,1,3} \\ I_{3,1,4} \end{cases} \implies \dim(\operatorname{Ker} d_2\partial_2/\operatorname{Ker} d_3) = \begin{cases} (4)_u \\ (3)_u \end{cases}.$$

Thus, the composition series of $K_{3,2}$ contains of simple comodules of dimensions $(1)_u + (2)_u$, $(2)_u + (3)_u$, $(3)_u$, $(4)_u$. On the other hand, we have

$$I_{1,1,1} \otimes I^*_{2,0,0} \otimes I^{*\otimes -1}_{1,0,0} = I_{1,1,1} \otimes I^*_{3,0,0} + I_{1,1,1} \otimes I_{2,1,0}.$$

According to (2.9), $I_{1,1,1} \otimes I^*_{3,0,0}$ contains the comodule $I_{1,-3,2}$. The dimension of $I_{1,-3,2}$ is $((2)_u + 2)(4)_u$ and the dimension of $I_{1,0,0}$ is $(1)_u + (2)_u$. Comparing the dimension, we get a contradiction, which means $\operatorname{Im} d_2 = \operatorname{Ker} d_3$. Thus, the complex is exact at the term $K_{3,2}$.

Moreover, from the above discussion we also have $\operatorname{Im} d_3/I_{1,-1,1} = I_{1,0,0}$. We wish to find $\operatorname{Ker} d_2\partial_2/\operatorname{Ker} d_3$ which will yield the composition series of $K_{3,2}$. We have

$$\operatorname{Ker} d_2\partial_2/\operatorname{Ker} d_3 \otimes I_{3,3,1} = I_{4,1,3} + I_{3,1,4}.$$

We notice that a similar equation holds for $I_{1,-2,2}$ and that this comodule is contained in $K_{4,3}$. Using the method of Corollary 3.2, we conclude that these comodules are in fact isomorphic. Thus, the composition series of $K_{3,2}$ contains $I_{1,0,0}$, two copies of $I_{1,-1,1}$, and $I_{1,-2,-2}$.

The above proof can be repeated to prove the exactness of the complex at any term $K_{k+1,k}$, $k \geqslant 3$ and to find the composition series of $K_{k+1,k}$. $\square$

4. Classification of simple comodules

In this section, we will construct for each triple (m, n, p), $m, n, p \in \mathbb{Z}$, $m \geqslant n$, a simple H_R-comodule. We find a condition for each of them to be splitting and compute their dimensions. Moreover, we prove that these simple comodules furnish all simple H_R-comodules up to isomorphism.

In the previous sections we have already constructed simple comodules for certain triples, namely those that correspond to partitions (i.e. satisfy condition in (2.1), the triples of the form $(1, -l, k)$; $l \neq k \geqslant 0$) (cf. (2.8)), and the triple $(1, 1, -1)$. Notice that $I_{1,1,-1}$ is invertible and for triples satisfying $m \geqslant n \geqslant 1$, $p \geqslant 1$, we have (cf. (3.1)) $I_{m,n,p} \otimes I_{1,1,-1} = I_{m+1,n+1,p-1}$. To define other simple comodules, we first set

$$I_{m,n,p} := I_{m+p,n+p,0} \otimes I_{1,1,-1}^{*\otimes p}$$

to reduce the problem to defining $I_{m,n,0}$.

For $m \geqslant n \geqslant 0$, $I_{m,n,0}$ is already defined. For $0 > m \geqslant n$, we set

$$I_{m,n,0} := I^*_{-n,-m,0}.$$

For $m > 0 > n$, we set

$$I_{m,n,0} := I_{1,n-m+1,m-1} \otimes I_{1,1,-1}^{\otimes m-1}.$$

It is clear that all the above comodules are simple. Since $\dim I_{1,1,-1} = 1$, we have the following formula for the dimension:

$$\dim I_{k,l,0} = \begin{cases} \big((2)_u + 2\big)(k - l + 1)_u, & \text{if } k \geqslant l \geqslant 1, \\ (k)_u + (k+1)_u, & \text{if } k > l = 0, \\ \big((2)_u + 2\big)(k - l)_u, & \text{if } k > 0 > l. \end{cases} \tag{4.1}$$

Proposition 4.1. *The simple H_R-comodule $I_{m,n,p}$ is splitting if and only if $(m + p)(n + p) \neq 0$.*

Proof. Since $I_{m,n,p} = I_{m+p,n+p,0} \otimes I_{1,1,-1}^{*\otimes p}$, $I_{m,n,p}$ is splitting if and only if $I_{m+p,n+p,0}$ is splitting.

If $m + p = 0$ then $n + p \leqslant 0$ and $I_{m+p,n+p,0} = \mathsf{S}^*_{-(n+p)}$ is not splitting. Analogously, if $n + p = 0$ then $I_{m+p,n+p,0} = \mathsf{S}_{m+p}$ is not splitting.

If $m + p \geqslant n + p > 0$ (respectively $0 > m + p \geqslant n + p$) then, according to Lemma 2.3, $I_{m+p,n+p,0}$ (respectively $I_{-(n+p),-(m+p),0}$) is splitting. If $m + p > 0 > n + p$ then $I_{m+p,n+p,0} = I_{1,n-m+1,m+p-1} \otimes I_{1,1,-1}^{*\otimes m+p-1}$ is splitting, according to Lemma 2.4. $\square$

Lemma 4.2. *Let (m, n, p), (x, y, z) be triples that correspond to partitions. Then $I_{m,n,p} \otimes I_{1,1,-1}^{\otimes t} \cong I_{x,y,z}$ if and only if $m + t = x$, $n + t = y$, $z + t = p$.*

Proof. Assume that there exists $t \geqslant 0$ such that

$$I_{m,n,p} \otimes I_{1,1,-1}^{\otimes t} \cong I_{x,y,z}. \tag{4.2}$$

We can assume that $p = z = 0$.

Consider first the case $x > y$, according to (2.4) we have

$$I_{x,y,0} \otimes I_{1,1,t+1} = I_{x+1,y+1,t+1} + I_{x+1,y,t+2} + I_{x,y+1,t+2} + I_{x,y,t+3}.$$

A similar equation holds for $I_{m,n,0} \otimes I_{1,1,t+1}$ (in particular $m > n$). Therefore (4.2) is equivalent to

$$\begin{aligned} &I_{m+t+1,n+t+1,1} + I_{m+t+1,n+t,2} + I_{m+t,n+t+1,2} + I_{m+t,n+t,3} \\ &\quad = I_{x+1,y+1,t+1} + I_{x+1,y,t+2} + I_{x,y+1,t+2} + I_{x,y,t+3}. \end{aligned}$$

Notice that, since $u > 0$, $(m)_u = (n)_u$ iff $m = n$. Thus, by comparing dimensions we conclude that

$$I_{m+t+1,n+t+1,1} \cong I_{x+1,y+1,t+1}, \tag{4.3}$$

which impls $t = 0$. The case $m = n$ is treated similarly. $\square$

Theorem 4.3. *Given triples (m, n, p), $(x, y, z) \in \mathbb{Z}^3$, with $m \geqslant n$, $x \geqslant y$. If $(m, n, p) \neq (x, y, z)$, then the simple comodules $I_{m,n,p}$, $I_{x,y,z}$ are non-isomorphic.*

Proof. It is sufficient to show that $I_{m,n,0} \otimes I_{1,1,-1}^{\otimes t} = I_{x,y,0}$ implying $m = x$, $n = y$, $t = 0$.

Case 1 ($m \geqslant n \geqslant 0$). (a) If $x \geqslant y \geqslant 0$ then the assertion follows from Lemma 4.2.

(b) If $0 > x \geqslant y$ then $I_{x,y,0} = I^*_{-y,-x,0}$ and since $I_{m,n,0} \otimes I_{-y,-x,0}$ decomposes into the direct sum of simple comodules, none of which is isomorphic to $I_{1,1,-1}^{\otimes t}$ for its dimension is larger than 1, we have

$$\mathrm{Hom}\big(I_{m,n,0} \otimes I_{-y,-x,0}, I_{1,1,-1}^{\otimes t}\big) = 0.$$

That is $I_{m,n,0} \neq I_{x,y,0}$.

(c) If $x > 0 > y$, we show that $I_{m,n,0} \otimes I_{1,1,-1}^{\otimes t} \neq I_{x,y,0}$. Assume the contrary. By comparing dimensions, we have $m - n + 1 = x - y$. Since $I_{x,y,0} = I_{1,y-x+1,x-1} \otimes I_{1,1,-1}^{x-1}$, we have, according to (2.9),

$$\operatorname{Hom}\bigl(I_{m,n,0} \otimes I_{1,1,-1}^{\otimes s}, \wedge_{-y}^{*} \otimes \mathsf{S}_{x-y-1}\bigr) \neq 0$$

for $s = t - x - 1$ or, equivalently,

$$\operatorname{Hom}\bigl(\wedge_{-y} \otimes I_{m,n,0} \otimes I_{1,1,-1}^{\otimes s}, \mathsf{S}_{m-n}\bigr) \neq 0.$$

But it can be easily deduced from Lemma 4.2 that this is not the case.

Case 2 ($m > 0 > n$). There are two possibilities for x, y: $x > 0 > y$ and $0 \geqslant x \geqslant y$.

(a) If $0 \geqslant x \geqslant y$ then we can show analogously as in Case 1(b) that $I_{m,n,0} \otimes I_{1,1,-1}^{\otimes t} \neq I_{x,y,0}$ for any t.

(b) If $x > 0 > y$, the assumption implies that $m + n = x + y$. Since $I_{1,-l,k}$ is a direct summand of $\wedge_k \otimes \mathsf{S}_l^{*}$, we have for $s = t + m - x$:

$$\operatorname{Hom}\bigl(\wedge_{m+1} \otimes \mathsf{S}_{-n}^{*} \otimes I_{1,1,-1}^{\otimes s}, \wedge_{x+1} \otimes \mathsf{S}_{-y}^{*}\bigr) \neq 0.$$

Then according to (1.4), (1.5), we have

$$\begin{aligned} 0 &\neq \operatorname{Hom}^H\bigl(\mathsf{S}_{-n}^{*} \otimes \wedge_m \otimes I_{1,1,-1}^{\otimes s}, \mathsf{S}_{-y}^{*} \otimes \wedge_x\bigr) \\ &= \operatorname{Hom}^H\bigl(\mathsf{S}_{-y} \otimes \wedge_m \otimes I_{1,1,-1}^{\otimes s}, \mathsf{S}_{-n} \otimes \wedge_x\bigr). \end{aligned}$$

By comparing dimensions and using Lemma 4.2 we conclude that $s = 0$ and $m = x$, $n = y$.

Case 3 ($0 \geqslant m \geqslant n$). There is only one possibility for x, y: $0 \geqslant x \geqslant y$ and this case is treated similarly as Case 1(a). □

Corollary 4.4. *The following rule holds for dual comodules*:

$$I_{m,n,p}^{*} = I_{-n,-m,-p}. \tag{4.4}$$

Proof. It remains only to prove, for $m, n > 0$, $I_{m,-n,0}^{*} = I_{n,-m,0}$, or $I_{1,-m-n+1,m-1}^{*} = I_{1,-m-n+1,n-1} \otimes I_{1,1,-1}^{\otimes m+n-2}$. Let $k = m - 1$, $l = m + n - 1$. It then leads to showing that

$$I_{1,-l,k}^{*} = I_{1,-l,l-k-1} \otimes I_{1,1,-1}^{\otimes l-1}.$$

Now, according to Theorem 4.3 and (2.9), $I_{1,-l,k}$ is the unique comodule that is a direct summand of $\wedge_{l-k+1} \otimes \mathsf{S}_l^*$ and $\wedge_{l-k+2} \otimes \mathsf{S}_{l+1}^*$, but not of $\wedge_{l-k} \otimes \mathsf{S}_{l-1}^*$. Thus, using various isomorphisms, we conclude that $I_{1,-l,k}^*$ is a subcomodule of $\wedge_{l-k} \otimes \mathsf{S}_l^* \otimes I_{1,1,-1}^{\otimes l-1}$ and $\wedge_{l-k+1} \otimes \mathsf{S}_{l+1}^* \otimes I_{1,1,-1}^{\otimes l-1}$. Therefore $I_{1,-l,k}^* = I_{1,-l,l-k-1} \otimes I_{1,1,-1}^{\otimes l-1}$. $\square$

4.1. The completeness of the set $\{I_{m,n,p}\colon m \geqslant n,\, m, n, p \in \mathbb{Z}\}$

We have so far constructed a set of simple comodules $\{I_{m,n,p}\colon m, n, p \in \mathbb{Z},\ m \geqslant n\}$. In the next step, we exhibit some formulas for the tensor product of these comodules with $I_{1,0,0}^*$ and deduce from these formulas that these comodules furnish all simple H_R-comodules.

Lemma 4.5. *The composition series of $I_{m,1,0} \otimes I_{1,0,0}^*$, $m \geqslant 2$, consists of five components: $I_{m-1,1,0}, I_{m,1,-1},\ I_{m,-1,1}$ and two copies of $I_{m,0,0}$.*

Proof. We use induction. For $m = 2$, we know that the composition series of $I_{1,1,0} \otimes I_{1,0,0}^*$ consists of $I_{1,1,-1},\ I_{1,-1,1}$ and two copies of $I_{1,0,0}$. From the equations

$$I_{2,1,0} \otimes I_{1,0,0}^* = I_{1,1,0} \otimes I_{1,0,0} \otimes I_{1,0,0}^* - I_{1,1,1} \otimes I_{1,0,0}^*,$$
$$I_{1,1,1} \otimes I_{1,0,0}^* = I_{1,1,0} + I_{1,0,1},$$

we see that $I_{2,1,0} \otimes I_{1,0,0}^*$ contains $I_{1,1,0},\ I_{2,1,-1},\ I_{2,-1,1}$ and two copies of $I_{2,0,0}$ in its composition series.

For $m \geqslant 3$, we have $I_{m-1,0,0} \otimes I_{1,1,0} = I_{m,1,0} + I_{m-1,1,1}$, hence

$$\begin{aligned} I_{m,1,0} \otimes I_{1,0,0}^* &= I_{m-1,0,0} \otimes I_{1,1,0} \otimes I_{1,0,0}^* - I_{m-1,1,1} \otimes I_{1,0,0}^* \\ &= I_{m-1,0,0} \otimes I_{1,1,0} \otimes I_{1,0,0}^* - (I_{m-2,1,1} + I_{m-1,0,1} + I_{m-1,1,0}). \end{aligned}$$

By the induction hypothesis, $I_{m-1,0,0} \otimes I_{1,1,0} \otimes I_{1,0,0}^*$ contains $I_{m,1,-1}, I_{m,-1,1}, I_{m-1,0,1}, I_{m-2,1,1}$, any two copies of $I_{m,0,0},\ I_{m-1,1,0}$, each in its composition series. Thus $I_{m,1,0} \otimes I_{1,0,0}^*$ contains $I_{m-1,1,0},\ I_{m,1,-1},\ I_{m,-1,1}$ and two copies of $I_{m,0,0}$ in its composition series. $\square$

Corollary 4.6. *For $n \geqslant 2$, the composition series of $I_{1,-n,0} \otimes I_{1,0,0}^*$ consists of $I_{1,-n-1,0}$, $I_{-1,-n,1}$, $I_{1,-n,-1}$ and two copies of $I_{0,-n,0}$.*

Proof. Multiplying the equation $I_{1,0,0} \otimes I_{n,0,0}^* = I_{n-1,0,0}^* + I_{1,-n,0}$ with $I_{1,0,0}^*$, we get

$$\begin{aligned} I_{1,-n,0} \otimes I_{1,0,0}^* &= I_{1,0,0} \otimes (I_{n,0,0} \otimes I_{1,0,0})^* - I_{n-1,0,0}^* \otimes I_{1,0,0}^* \\ &= I_{1,0,0} \otimes I_{n,1,0}^* + I_{1,0,0} \otimes I_{n+1,0,0}^* - I_{n,0,0}^* - I_{n-1,1,0}^*. \end{aligned}$$

According to Lemma 4.5, $I_{n,1,0} \otimes I_{1,0,0}^*$ contains $I_{n-1,1,0},\ I_{n,1,-1},\ I_{n,-1,1}$ and two copies of $I_{n,0,0}$ in its composition series, hence, by means of (4.4), the composition series of $I_{n,1,0}^* \otimes I_{1,0,0}$ consists of $I_{-1,1-n,0},\ I_{-1,-n,1},\ I_{1,-n,-1}$ and two copies of $I_{0,-n,0}$.

According to (2.9), $I_{1,0,0} \otimes I^*_{n+1,0,0} = I_{0,-n,0} + I_{1,-n-1,0}$. Thus, we conclude that the composition series $I_{1,-n,0} \otimes I^*_{1,0,0}$ consists of $I_{-1,-n+1,0}$, $I_{-1,-n,1}$, $I_{1,-n,-1}$ and two copies of $I_{0,-n,0}$. □

Lemma 4.7. *For $m \geqslant 2$, $n \geqslant 1$, the following decomposition holds*:

$$I_{m,-n,0} \otimes I^*_{1,0,0} = I_{m-1,-n,0} + I_{m,-n-1,0} + I_{m,-n,-1}. \tag{4.5}$$

Proof. We first prove that, for $k, l \geqslant 1$,

$$\mathrm{Hom}\big(I_{1,-l,k} \otimes I^*_{1,0,0}, I_{1,-l-1,k}\big) \neq 0, \quad \mathrm{Hom}\big(I_{1,-l-1,k}, I_{1,-l,k} \otimes I^*_{1,0,0}\big) \neq 0. \tag{4.6}$$

Indeed, consider the diagram

$$\begin{array}{ccc} I_{1,1,k-1} \otimes I^*_{l,0,0} \otimes I^*_{1,0,0} & \xrightarrow{d_{k+1,l} \otimes \mathrm{id}} & I_{1,1,k} \otimes I^*_{l+1,0,0} \otimes I^*_{1,0,0} \\ {\scriptstyle \mathrm{id} \otimes X^*_{l+1}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \otimes X^*_{l+2}} \\ I_{1,1,k-1} \otimes I_{m+n,0,0} & \xrightarrow{d_{k+1,l+1}} & I_{1,1,k+1} \otimes I_{l+2,0,0} \end{array}$$

which can be easily shown to be commutative. Since the vertical maps are surjective, the induced map $\mathrm{Im}(d_{k+1,l} \otimes \mathrm{id}) \to \mathrm{Im}\, d_{k+1,l+1}$ is non-zero, proving the first equation in (4.6). Since $I_{1,-l-1,k}$ is splitting, this equation implies the second one.

Now, using (4.6) we can easily show that the comodules on the right-hand side of (4.5) are subcomodules of that on the left-hand side. Then comparing dimensions we derive the equation. □

Theorem 4.8. *The set $\{I_{m,n,p}\colon m \geqslant n,\ m, n, p \in \mathbb{Z}\}$ furnishes all simple H_R-comodules.*

Proof. Each simple H_R-comodule is a subquotient of H_R and $H_R = T(V^* \otimes V)/I$, with $T(V^* \otimes V)$, I being H_R-comodules. On the other hand,

$$T(V^* \otimes V) = \bigoplus_{i \geqslant 0}^{\infty} (V^* \otimes V)^{\otimes i}.$$

Therefore it is sufficient to prove that any comodule contained in the composition series of $(V^* \otimes V)^{\otimes i}$ is isomorphic to one of the constructed comodules.

Since the similar assertion holds for $V^{\otimes i}$, it is sufficient to prove that the composition series of $I_{m,n,p} \otimes V^* = I_{m,n,p} \otimes I^*_{1,0,0}$ contains only simple comodules isomorphic to the constructed simple comodules.

For $m \geqslant n \geqslant 2$, by multiplying (2.6) with $I_{1,1,-1}$, we obtain

$$I_{m,n,0} \otimes I^*_{1,0,0} = \begin{cases} I_{m-1,n,0} + I_{m,n-1,0} + I_{m,n,-1}, & \text{if } m > n, \\ I_{m,m-1,0} + I_{m,m,-1}, & \text{if } m = n. \end{cases}$$

For $m > n = 1$, according to Lemma 4.5, the composition series of $I_{m,1,0} \otimes I^*_{1,0,0}$ consists of five components: $I_{m-1,1,0}$, $I_{m,1,-1}$, $I_{m,-1,1}$ and two copies of $I_{m,0,0}$.

For $m = n = 1$, according to Corollary 3.2, $I_{1,1,0} \otimes I^*_{1,0,0}$ contains $I_{1,1,-1}$, $I_{1,-1,1}$, and two copies of $I_{1,0,0}$ in its composition series.

For $m > n = 0$, we have, by dualizing (2.9):

$$I_{m,0,0} \otimes I^*_{1,0,0} = I_{m-1,0,0} + I_{m,-1,0}.$$

For $m = 1,\ -1 \geqslant n$, according to Corollary 4.6, $I_{1,n,0} \otimes I^*_{1,0,0}$ contains $I_{1,n-1,0}$, $I_{-1,n,1}$, $I_{1,n,-1}$ and two copies of $I_{0,n,0}$ in its composition series.

For $m \geqslant 2,\ -1 \geqslant n,$ according to (4.5), we have

$$I_{m,n,0} \otimes I^*_{1,0,0} = I_{m-1,n,0} + I_{m,n-1,0} + I_{m,n,-1}.$$

For $0 \geqslant m \geqslant n$, we have, dualizing (2.3):

$$I_{m,n,0} \otimes I^*_{1,0,0} = \begin{cases} I_{m,n-1,0} + I_{m-1,n,0} + I_{m,n,-1}, & \text{if } m > n, \\ I_{m,m-1,0} + I_{m,m,-1}, & \text{if } m = n. \end{cases}$$

Thus, in all cases, the composition series of $I_{m,n,p} \otimes I^*_{1,0,0}$ with $m \geqslant n$; $m, n, p \in \mathbb{Z}$, contain only known simple comodules. The proof is complete. $\square$

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